VISCOUS FLOW THEORY

LECTURE 8

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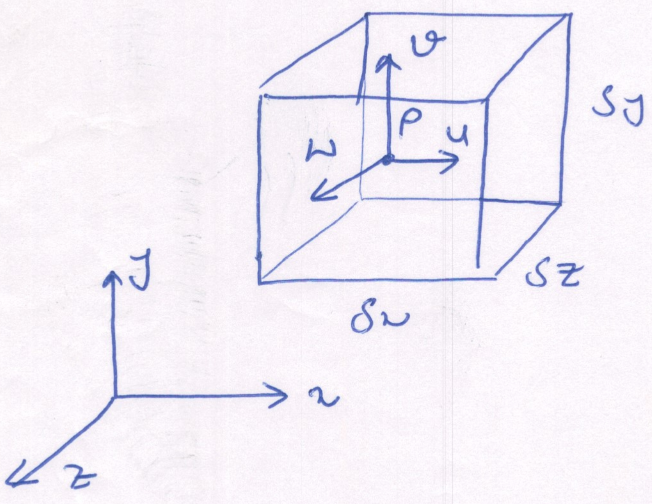
Differential form of Continuity Equation

→ applied to an infinitesimal control volume

→ Now take our CV to be small stationary cubical element

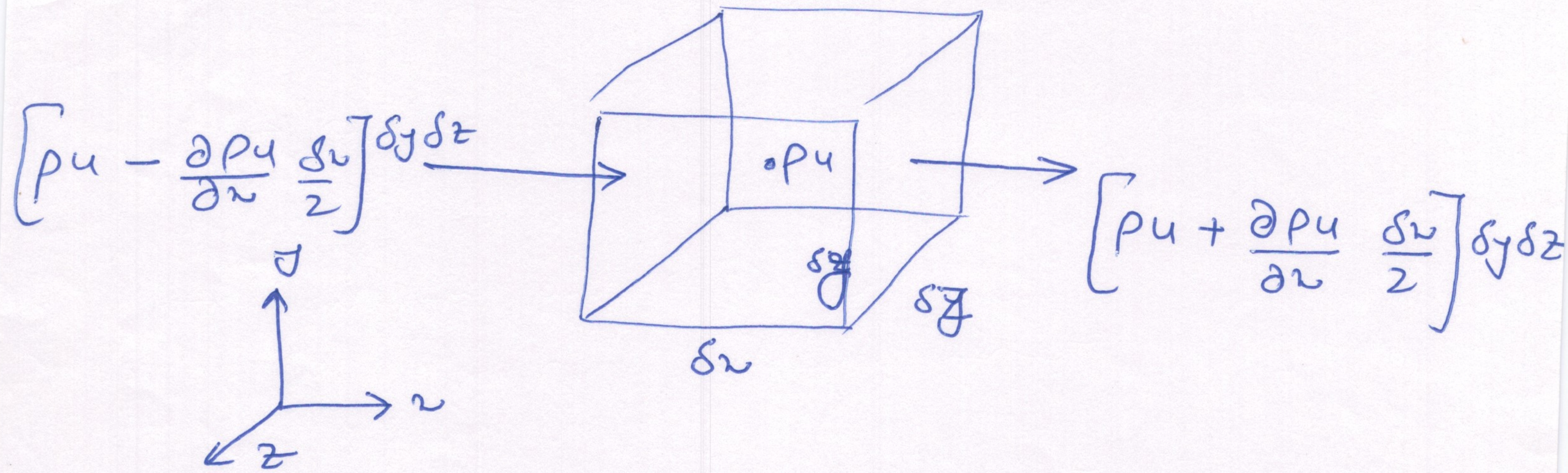
→ at the center of the element the fluid density is ρ

→ velocity has u, v, w } components



for small element

$$\frac{\partial}{\partial t}\int_{CV}\rho \, d\forall \approx \frac{\partial \rho}{\partial t}\,\delta x\,\delta y\,\delta z$$



Rate of mass flow thru the surfaces of the element can be obtained by considering flow in each of coordinate directions separately

x direction

Net rate of mass out flow in x direction

$$= \left[\rho u + \frac{\partial \rho u}{\partial x}\frac{\delta x}{2}\right]\delta y\,\delta z$$

$$- \left[\rho u - \frac{\partial \rho u}{\partial x}\frac{\delta x}{2}\right]\delta y\,\delta z$$

$$= \frac{\partial(\rho u)}{\partial x}\,\delta x\,\delta y\,\delta z$$

Net rate of mass out flow in y direction

$$= \frac{\partial(\rho v)}{\partial y}\,\delta x\,\delta y\,\delta z$$

Net rate of mass out flow in z - direction

Net rate of mass outflow $= \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right] \delta x \delta y \delta z$

Integral eqn

$$\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int \rho \vec{V} \cdot \hat{n} \, dA = 0$$

$$\frac{\partial \rho}{\partial t} \delta x \delta y \delta z + \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right] \delta x \delta y \delta z = 0$$

$$\frac{\partial \rho}{\partial t} + \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right] = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{V} = 0}$$

Two special cases of interest

→ for steady flow of compressible fluid

$$\vec{\nabla} \cdot \rho \vec{V} = 0$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

→ ρ not a function of time
but function of position

→ for incompressible fluid
ρ is constant throughout
the flow field

$$\vec{\nabla} \cdot \vec{V} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

{ for both steady / unsteady flow
of incompressible fluids)

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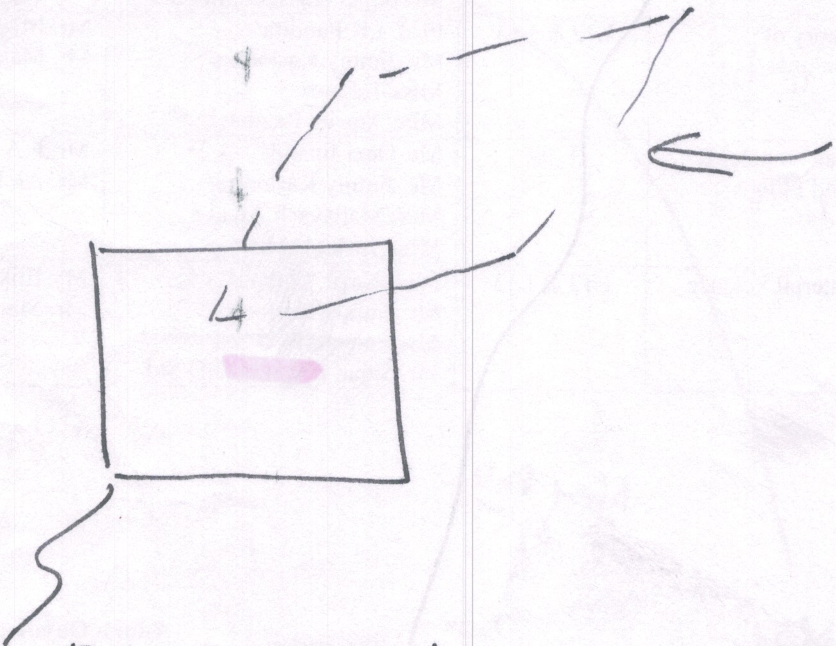
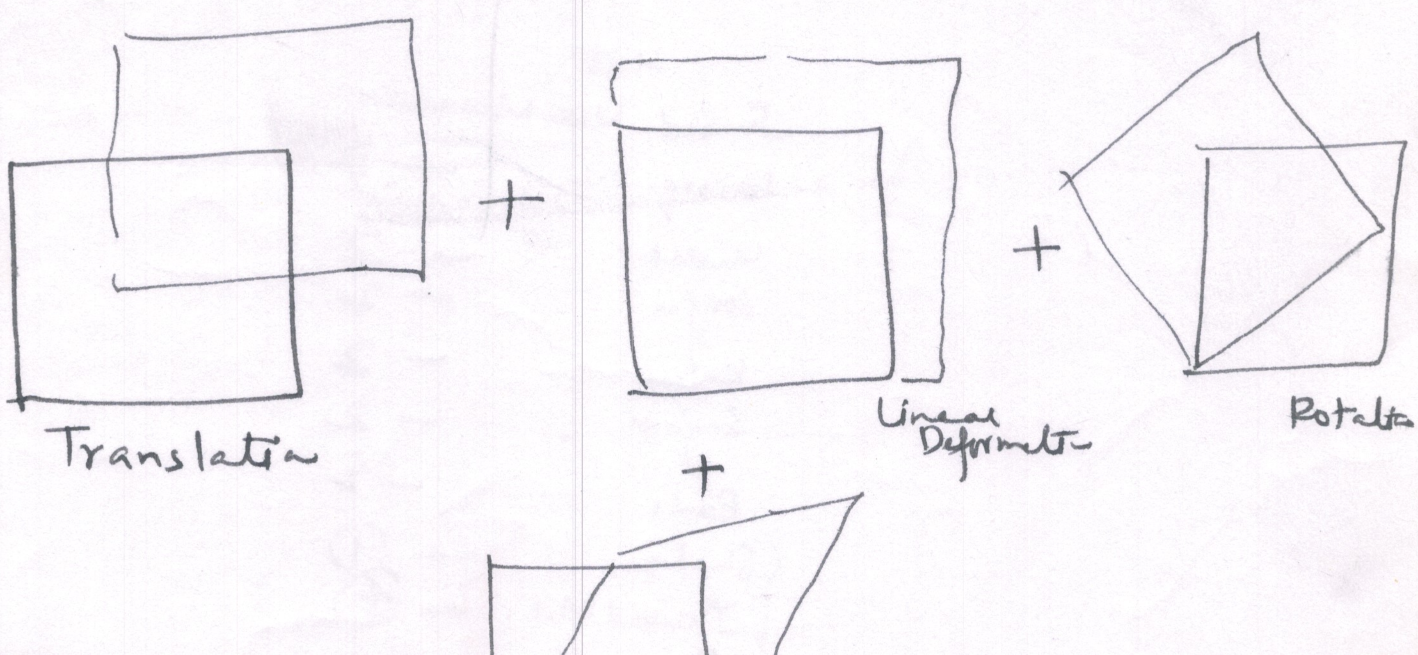
Fluid Element Kinematics

→ A small fluid element in the shape of the cube is initially in one position will move to another position during a short time interval δt



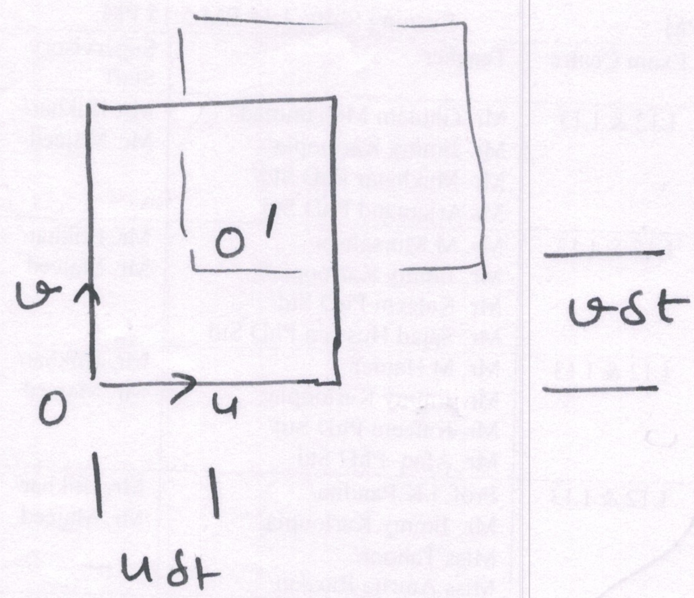
Element at $t_0 + \delta t$

Element at t_0



→ Although these movements take place simultaneously we consider each one separately

Linear Motion & Deformation



→ Simplest of all

→ in a small interval δt

a particle located at O will move to

point O'

→ if all the points in the element have

same velocity

(in absence of velocity gradients)

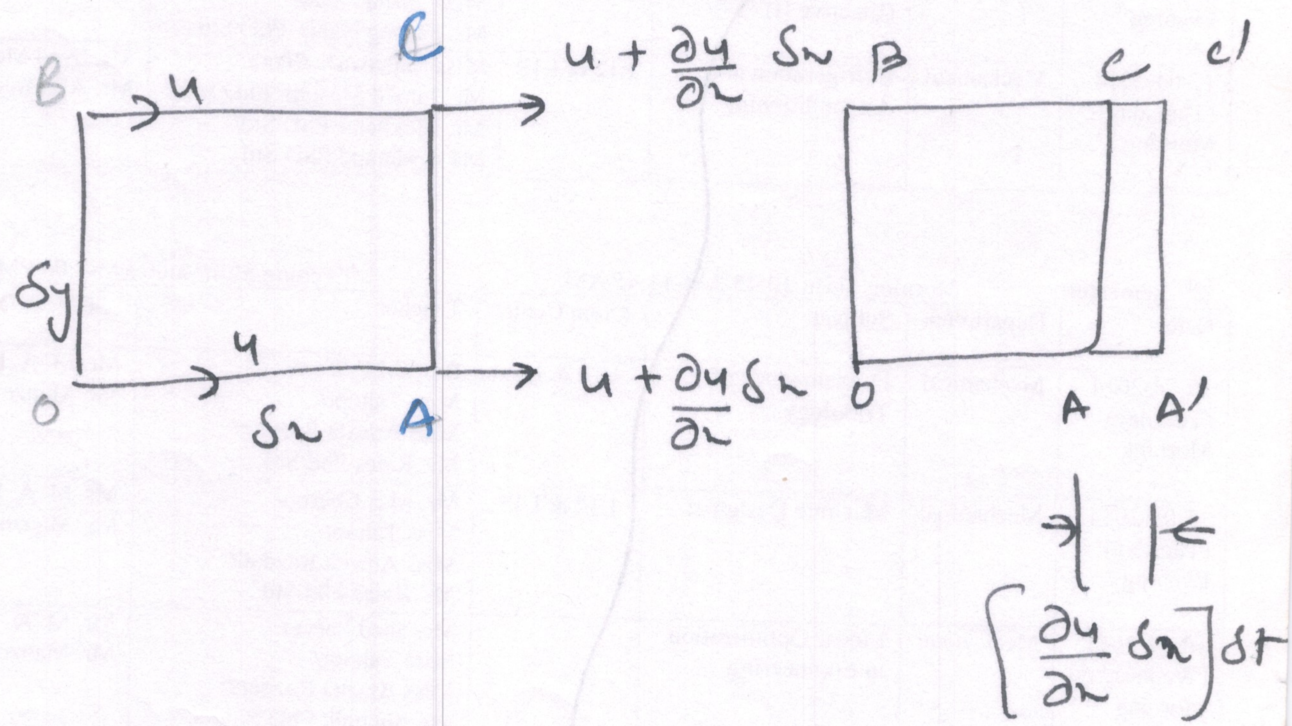
the elements will translate

→ In presence of velocity gradients

→ the elements will generally be deformed & rotated as it moves

→ Consider the effect of single velocity gradient $\frac{\partial u}{\partial x}$ on small cube

$\delta x, \delta y, \delta z$



→ x component of velocity at O & B is u

→ at nearby pts A & C

$$u + \left(\frac{\partial u}{\partial x}\right)\delta x$$

difference in velocity causes stretching line OA stretches to OA'

& line BC 〃 to BC'

$$\text{Change in } \delta\forall = \left(\frac{\partial u}{\partial x}\right)\delta x\,\delta t\;\delta y\,\delta z$$

Rate at which the volume $\delta\forall$ is changing per unit volume due to velocity gradient $\frac{\partial u}{\partial x}$

$$\frac{1}{\delta\forall}\,\frac{d(\delta\forall)}{dt} = \lim_{\delta t \to 0}\left[\frac{\left(\frac{\partial u}{\partial x}\right)\delta t}{\delta t}\right] = \frac{\partial u}{\partial x}$$

If velocity gradients $\frac{\partial v}{\partial y}$ & $\frac{\partial w}{\partial z}$ are present

$$\frac{1}{\delta\forall}\,\frac{d}{dt}(\delta\forall) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$= \vec{\nabla}\cdot\vec{V}$$

Volumetric

Dilatation rate

→ We see volume of the fluid change as element moves from one location to another in the flow field

→ for incompressible fluid

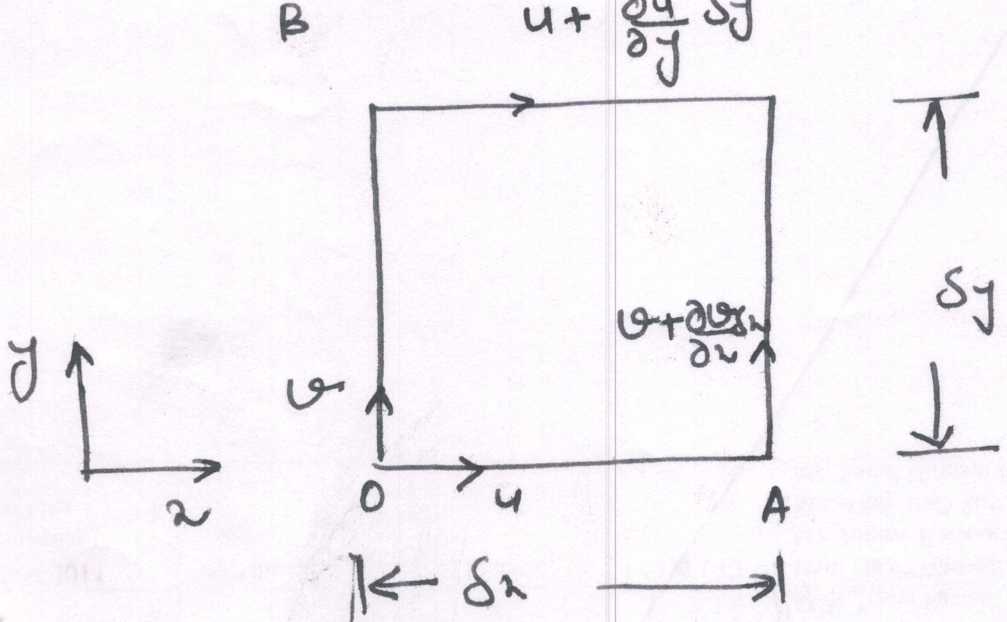
→ the volumetric dilation rate is zero since the element volume can't change without change in density

→ variations in $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}$ and $\frac{\partial w}{\partial z}$ simply cause a linear deformation in the sense that the shape of the element does not change.

Angular Motion & Deformation

→ We consider motion in x-y plane

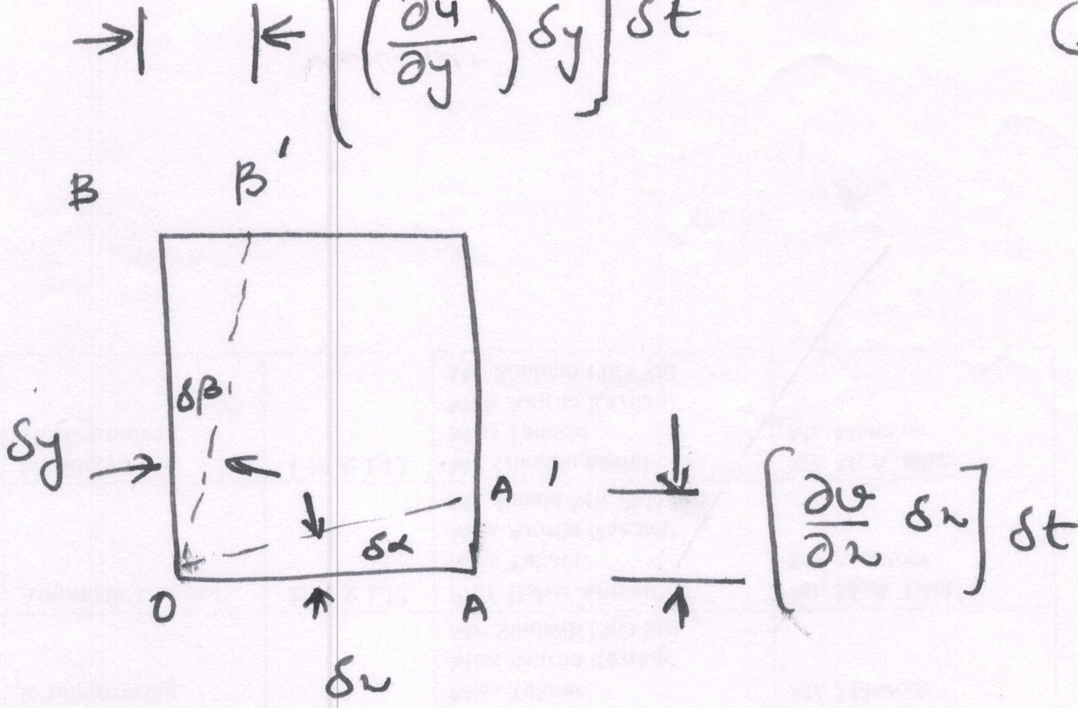
→ Velocity variation that causes rotation & angular deformation



After short interval time δt

→ the line segments OA & OB will rotate thru the angles $\delta\alpha$ & $\delta\beta$ to the new position

OA' & OB'



Angular velocity of line OA,

$$\omega_{OA} = \lim_{\delta t \to 0} \frac{\delta\alpha}{\delta t}$$

for small angles

$$\tan\delta\alpha \simeq \delta\alpha = \frac{\left[\frac{\partial v}{\partial x}\,\delta x\right]\delta t}{\delta x}$$

$$= \left(\frac{\partial v}{\partial x}\right)\delta t$$

so that

$$\omega_{OA} = \lim_{\delta t \to 0} \frac{\left[\frac{\partial v}{\partial x}\right]\delta t}{\delta t}$$

if $\frac{\partial v}{\partial x}$ is +ve, ω_{OA} will be counterclockwise

Similarly,

$$\omega_{OB} = \lim_{\delta t \to 0} \frac{\delta \beta}{\delta t}$$

$$\tan \delta\beta \simeq \delta\beta = \frac{\left(\frac{\partial u}{\partial y}\right)\delta y\, \delta t}{\delta y}$$

$$= \frac{\partial u}{\partial y}\,\delta t$$

$$\omega_{OB} = \frac{\partial u}{\partial y}$$

if $\frac{\partial u}{\partial y}$ is +ve, ω_{OB} is clockwise

$$\omega_z = \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]$$

average of ω_{OA} & ω_{OB}

Similarly,

$$\omega_x = \frac{1}{2}\left[\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right]$$

$$\omega_y = \frac{1}{2}\left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right]$$

$$\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

$$\vec{\omega} = \frac{1}{2}\,\mathrm{curl}\,\vec{V} = \frac{1}{2}\vec{\nabla}\times\vec{V}$$

$$\frac{1}{2}\vec{\nabla}\times\vec{V} = \frac{1}{2}\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j}$$

$$+ \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

$$\text{Vorticity} = \zeta = 2\vec{\omega} = \vec{\nabla}\times\vec{V}$$

We observe that the fluid element will rotate about the z_axis as an undeformed block

if

$$\omega_{OA} = -\omega_{OB}$$

when $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

→ otherwise the rotation will be associated with an angular deformation

→ When $\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x}$

→ rotation around the z_axis is zero

More generally,

$$\vec{\nabla} \times \vec{V} = 0$$

Irrotational

→ In addition to the rotation associated with the derivatives $\frac{\partial u}{\partial y}$ & $\frac{\partial v}{\partial x}$

→ these derivatives can cause angular deformation

→ Change in the original right angle formed by the lines OA & OB is termed as shearing strain, $\delta\gamma$

$$\delta\gamma = \delta\alpha + \delta\beta$$

→ Rate of shearing strain

$$\dot{\gamma} = \lim_{\delta t \to 0} \frac{\delta\gamma}{\delta t} = \lim_{\delta t \to 0} \left[\frac{\left(\partial v/\partial x\right)\delta t + \left(\frac{\partial u}{\partial y}\right)\delta t}{\delta t} \right]$$

$$\dot{\gamma} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\Rightarrow \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Angular deformation is zero